Distance geometry on

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# Distance and the triangle inequality

So far we have discussed only the incidence structure of the projective plane. We now introduce a distance function.

Definition. For P and Q in P2 define d(P, Q) = cos-ll<x, y)l'

where x and y are points of S2 and = P, = Q.

Remark: Because of the absolute value sign, the distance is well-defined. The distance between {x, —x} and {y, —y} is the spherical distance between the closest representatives. Also, all distances are <T/2.

Theorem 1. If P, Q, and R are points of P2 , then

d(P, Q) + d(Q, R) d(P, R).

Proof: Let r, p, and q be the respective distances. Choose representatives

P', Q', and R' so that R') O and (Q', R') O. As in Theorem 4.8,

IP' x Q'l = sin r, IR' x Q' I = sin p,

IP' x Q'IIR' x Q'l (P' x Q', Q' x R')



If Q') O, then we get the inequalities

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| sin r sin p | cos p cos r — cos q, |
| cos q | cos p cos r — sin r sin p, |
| cos q | cos (p + r), |

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Equality in this case would imply that



This means that P', Q', and R' are collinear on S2 and, hence, that P, Q, and R are collinear on P2 .

If (P', Q') < O, then use the Cauchy—Schwarz inequality in the form

# IP' x Q'IIR' x Q'! R') - P'), (6.1)

which yields

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| sin r sin p | cos q + cos p cos r, |
| cos (T — q) | cos(p + r), |



But q < q/2, so that q < — q < p + r. Equality can occur in this case only if q = IT/2 < p + r. As before, P', Q', and R' will be collinear. 

Remark: In this proof we have shown not only that the triangle inequality holds on P2 but also the familiar notion that equality cannot occur unless the three points in question are collinear. However, something unfamiliar also pops up here. Not every triple of collinear points satisfies the equality.

The situation is as follows.

Theorem 2. Three points of P2 are collinear if and only if they can be named P, Q, and R in such a way that either

i. d(P, Q) + d(Q, R) = d(P, R) ii. d(P, Q) + d(Q, R) + d(P, R) = T.

Proof: Suppose that we are given three collinear points for which (i) does not hold. Let e3 be a pole of the line of S2 determined by the three given points, and let el be a representative of one of the points, say P. Then we may choose e and with O < e, d) < T//2 such that the other two points are

1. = 4)e1 + (sin

and

1. = O)el ± (sin 0)e2),

where {el, e2, e3} is an orthonormal basis.

## Now d(P, cos-I (cos 4) = 4,

cos-I (COS 0)

and d(Q, cos-llcos@ ± e)l•

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| The minus sign cannot occur because it would imply that d(Q, R)  — 01, and, hence, an equation of the form (i) would• be satisfied. Similarly, if + + e < IT/2, we would have d(Q, R) = + 0, another version of (i). Thus, we must conclude that T/2 < + 0 < T, so that  d(Q, R) =  + 0)) = cos-l (costr — (e +      - (e + 4) = - d(P, Q) - d(P, R).  Conversely, if (i) holds, we showed in the proof of Theorem 1 that the points must be collinear. If (ii) holds, we have p + r = — q, so that cos(ff — q) = cos(p + r). By the same algebra as in Theorem 1, (6.1) becomes an equality, and the three points are collinear.  Isometries  Definition. A map T: P2 -+ P2 is called an isometry if d(P, Q) d(TP, TQ) for all P and Q in p2 .  Theorem 3. Let T be an isometry. If P, Q, and R are collinear, then TP, TQ, and TR are collinear.  Proof: Let P, Q, and R be collinear points. Let P' be the unique point on this line such that d(P, P') = T/2. Then  d(P, Q) + d(Q, P') = d(P, P') =  2  Hence,  d(TP, TQ) + d(TQ, TP') = d(TP, TP') =  By the previous theorem, TQ must lie on the line determined by TP and TP'. Similarly, TR must lie on this line.  The isometries of P2 are closely related to the isometries of S2  Theorem 4. Let T: P2 -+ P2 be an isometry. Then there exists a unique A e SO(3) such that T = A.  Proof: Choose el, e2, and e3 on S2 such that  TITEi trei for each i.  Then d(TTEi, TITEj) = d(1TEi, ITEJ) = cos | Isometries  143 |

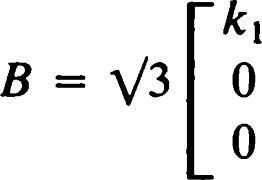
But d(trei, aej) = cos -l l<ea, and so E — I(et., ej)l. If i then

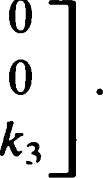
Kei, ej) = O. Otherwise, (q, ej) = 1. Thus, {et•} is an orthonormal basis of R3 . Let A be the orthogonal matrix such that Ati = ei for each i. Then A is an isometry of P2 , and A -I T leaves TCI, ITE2, and fixed. Let

M = IT(EI + €2 + €3) and write

## Ä -I TM = + + k3E3),

where 1<1, k2, and k3 are some numbers, with k} + + k; = 1. We claim that the Ikil are all equal. To see this, note that d(Ä -l TM, Ä -l TTEi) — - cos -l lkil.

But d(M, ITEi) = cos -1 (1/V3) and, hence, Ikil = 1/1/3 for all i. Let o



o

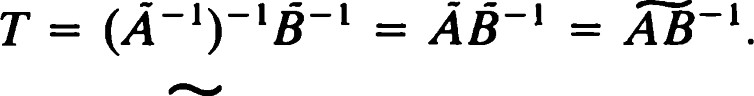
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Then

— for each i, and since each k? is 1/3, ÉÄ -I TM = M.

We will next prove that any isometry that leaves each Ei and IT(EI + €2 + €3) fixed must be the identity. Assuming this for the moment, we get

### ÉÄ -I T = 1; that is,



But B-1 = B. Hence, T = AB. Because AB is orthogonal, there is a unique member of SO(3) that determines the same isometry.

Theorem 5. If T is an isometry of P2 that leaves fixed each and T(EI + €2 + €3), then T is the identity.

Proof: Let us work in the homogeneous coordinate system determined by El, €2, and €3. Then (1, O, O), (O, 1, O), (O, O, 1), and (1, 1, 1) are fixed points. We first check that all points on the line joining (1, O, O) and (O, 1, O) are fixed. A typical such point is x = (cos a, sin a, O), where

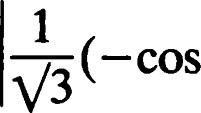
O < a < ff. Write Tx = (cos ß, sin ß, O), where O < ß < T. Then

d(TITE1, TX) = d(TE1, X), cos -l lcos ßl = cos -l lcos a ; that is, Icos ßl = Icos al. Thus, ß = a or ß = a. Motions If ß — a, we are finished. If ß = — a, then Tx = a, sin a, O).

Let M = (1, 1, 1). Then

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### cos -l a + sin a)

d(Tx, M) = cos —1  a + sin a) 

This is impossible unless a — T/2, in which case a = ß anyway. Similarly, we can show that all points on the sides of the triangle of reference A are fixed. Now each line of P2 contains at least two fixed points because it intersects A at least twice. Therefore, every line is fixed, and, hence, every point is a fixed point.

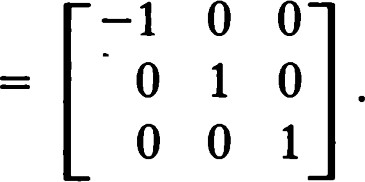
### Motions

Let C be a line of S2 . Then the reflection in the line ITC is the isometry of P2 defined by



Theorem 6. OTC leaves fixed every point on ITC and the pole of qt. No other points are fixed.

Proof: Choose an orthonormal basis of E3 with respect to which



Now Oex = x if and only if x lies on the line joining (O, 1, O) and (O, O, 1). Also Oex = —x if and only if x = (1, O, O) or (—1, O, 0). Thus, the fixed points of Oe are as claimed.

Let C be a line of P2 , and let be its pole. Then the product of two reflections where m and n pass through E, is called a rotation about E. Because line goes through if and only if it is perpendicular to e, we also call O„On a translation along e, and we call a glide reflection. If m -L n, then is called a half-turn.

Theorem 7. The fixed lines of a reflection are the line offixed points and all lines perpendicular to this line. 145 Proof: A line is fixed if and only if its pole is a fixed point.

Theorem 8. A rotation other than a half-turn or the identity has a unique fixed point and a unique fixed line. The point is the pole of the line.

Proof: Let Ä be a typical rotation of P2 , where A e SO(3). Then we will have solved the problem if we can find all those points x e S2 such that Ax = ±x. But this calculation was done in finding the fixed lines of a rotation of S2 . If x is a fixed point of A, then is the unique fixed point of A, and the line whose pole is ax is the unique fixed line of A.

Theorem 9.

i. Every reflection is a half-turn, and every half-turn is a reflection. ii. Every glide reflection is a rotation.

Corollary. Every isometry of P2 is a rotation.

Remark: It is easy to show that the three reflections theorem and the representation theorems for rotations and translations (Theorems 4.15, 16, 19, and 20) hold in P2 (Exercise 6).

### Elliptic geometry

The geometry of P2 is traditionally called elliptic geometry. So far, we have discussed its incidence properties, defined the notion of distance, and classified the isometries. We have seen that elliptic geometry is a simplification of spherical geometry.

Definition. A segment in P2 is a set of the form TO, where is a minor segment in S2 . The length of fib is the length of a. The end points of TO are the images by IT of the end points of o.

Theorem 10.

1. Each pair {A, B} of points in P2 is the end point set of two segments.

The union of these segments is the line AB, and their intersection is

1. For a segment of length L with end points A and B, we have d(A, B) = L if L €2. Otherwise, d(A, B) = - L.

Definition. A ray is a segment of length IT/2 with one end point removed. The remaining endpoint is called the origin of the ray.

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| Remark: The definitions of ray in Euclidean, spherical, and elliptic geometry may seem at first to have little in common. There is a unifying idea, however. Starting at the origin of the ray, we move in a particular direction as long as the path we have traced out is the shortest path to this origin. If this continues forever, as in the Euclidean case, the ray continues forever. On the sphere, however, once we reach the point antipodal to the ray's origin, we lose this uniqueness. In differential geometry the point where this happens is called a cut point. In P2 we reach a cut point at distance IT/2.  Theorem 11. Let P and Q be points with d(P, Q) < IT/2. Then there is a unique ray with origin P that contains Q. We denote this ray by PQ.  The definition of angle in elliptic geometry is the same as our previous definitions. The radian measure of an angle 4 PQR is determined by choosing a representative for Q, choosing the representatives for P and R closest to Q, and computing the radian measure of the spherical angle so determined.  The notion of half-plane does not occur in P2 . One can, however, define the interior of an angle.  A triangle in elliptic geometry is a figure of the form ITA, where A is a spherical triangle.  Theorem 12. If P, Q, and R are three noncollinear points ofP2 , there is a triangle having P, Q, and R vertices. The triangle is the union of three segments.  Remark: Our treatment of elliptic geometry has been brief. Most of the questions we have studied in Euclidean and spherical geometry have analogues that can be studied in the elliptic setting. Some of these are explored in the exercises.  EXERCISES  1. Find the distance d(P, Q), where P = (—1, O, 1) and Q = (1, 1, O) in homogeneous coordinates with respect to {El, €2, €3}.  Prove that every pair of distinct lines in P'2 has a common perpendicular. Only a slight modification of your proof of Theorem 4.10 is required.  ii. Find the common perpendicular to the lines + 2x2 = O and  2.X2 — X3 | Elliptic geometry  1 47 |

1. i. Prove the projective version of Theorem 4.11 concerning erecting and dropping perpendiculars.
   1. Is the foot of the perpendicular the point on C closest to P?
2. Let A: R3 -4 R3 be linear. Then define ÄTtx = TAX so that Ä maps

P2 -+ P2 . Under what conditions on A will Ä be an isometry? Illustrate using the matrix

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1. Prove Theorem 9 and its corollary.
2. Verify the remark following Theorem 9 (that the three reflections and representation theorems hold in P2).
3. i. Given three nonconcurrent lines a, ß, and -y, show how to find a point P and a line C such that OaOßOY = OeHp.
   1. If OeHp = O„HQ, what relationships must hold among e, m, P, and Q?
4. Let P and Q be distinct points. Find g ( {P,
5. Under what conditions will two rotations about distinct points commute?
6. Let P, Q, and R be mutually perpendicular points of S2 . Show that there are four isometries T of p2 that leave ITP, ITQ, and TR fixed. (Hint: Choose an appropriate orthonormal basis and compute the possible forms of the matrix of T.)
7. Find the symmetry group of the figure in P2 formed by two perpendicular lines.
8. Suppose that an isometry T of p2 has three concurrent fixed lines. Show that T must be a half-turn.
9. Prove Donkin's theorem: Let PQR be a triangle. Let a, ß, and py be rotations (translations) that take P to Q, Q to R, and R to P, respectively. Then pyßa is the identity.
10. Prove Theorem 10.
11. Prove Theorem 11.
12. Let P and Q be points with d(P, Q) < a/2. Prove that PQ n QP is the segment with end points P and Q and length d(P, Q).
13. Prove that the notion of radian measure for angles in P2 is welldefined.
14. What happens to Theorem 4.41 in p2?
15. i. Propose a definition for the perpendicular bisector of a segment in p2.

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|  | ii. Define the midpoint of a segment in such a way that each segment has a unique midpoint. | Elliptic geometry |
| 20. | Prove that there are exactly two reflections that interchange a given pair of lines in P2 . |  |
| 21. | Let e be a line of P2 , and let P and Q be any points not on e. Show that there is a segment joining P and Q that does not meet e. (This is why we do not attempt to define the notion of half-plane in P2 .) |  |
| 22. | Define the interior of an angle in P2 . Does the crossbar theorem hold? |  |
| 23. | Define the interior of a triangle in P2 . Show that P2 may be regarded as the union of four equilateral triangles (and their interiors). Each triangle should have three right angles. |  |
| 24. | Prove Theorem 12. Is the triangle unique? |  |
| 25. | Prove that if X is a point in the interior of a triangle A, there is a segment containing X whose end points are on A. |  |
| 26. | Prove that the perpendicular bisectors of the three sides of a triangle are concurrent. In light of Exercise 18, what further results can be obtained? |  |
| 27. | Prove that the congruence theorems for spherical triangles are valid in P2 as well (Theorems 55—57 of Chapter 4). |  |
| 28. | 1. Show that the finite groups of isometries of P2 may be identified with those listed in Theorem 4.58. 2. For each such group find a figure in P2 of which it is the symmetry group. |  |

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